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Inverse Kinematics of a Manipulator for Minimally Invasive Surgery

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1 Introduction

We describe the inverse kinematics of our robotic system for minimally invasive surgery. Special respect is given to an intuitive operability of the user interface. Therefore we apply the concept of so-called trocar kinematics. Meaning that the manipulator (in our case an instrument for minimally invasive surgery) has to pass a fixed opening through the surface of the patient's body. It is the principle idea of minimally invasive surgery to perform all surgical tasks through small keyhole-openings (so-called ports) in order to avoid traumatic invasions. Evidently, this procedure restricts the degrees of freedom of the instrument. Feed and rotation axes always have to intersect with the fixed port. With respect to kinematics this point is often called trocar point. Given the position and rotation of the end effector, as a result of the instrument's trocar kinematics, we get the position and rotation of the robot flange that bears the instrument. Therefore we have full cartesian control of the instruments. Since no direct cartesian control of the robot is possible within our interface, we also have to determine the inverse kinematics of our robots. As a final result we will get three joint angles for the minimally invasive instrument, while we get six angles for directly controlling the robot's joints.

1.1 Mechanical Setup

The motor part of our system consists chiefly of two instruments for minimally invasive surgery, which are mounted on Kuka industrial robots. Both instruments employed in our system are provided by Intuitive Surgical Inc. Each of them consists of several mechanical parts. The part mounted on the driving device is called drive box. It contains four driving wheels, one for each degree of freedom of the instrument. Note, that both fingers of the end effector can be moved independently. In our setup the wheels are directly driven by servos that are coupled to the device via an Oldham coupling. The driving wheels are connected to spindles that control the moving parts of the instrument via steel wire linkage. The long cylindrical part which catenates the drive box with the end effector is called shaft. It is constructed as a hollow carbon fibre tube that contains the steel

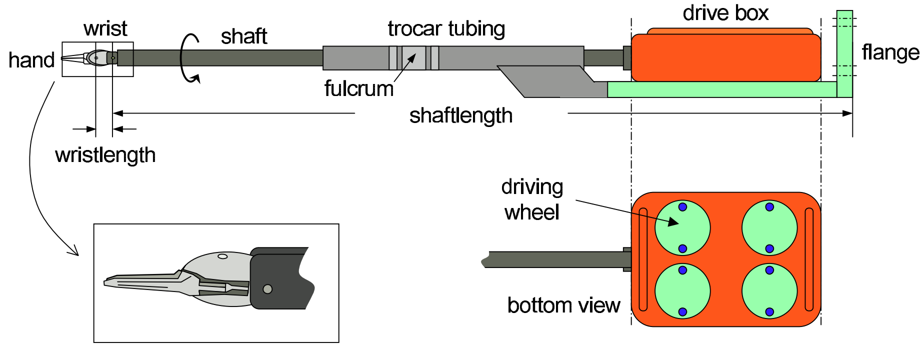

Fig. 1. Instrument for Minimally Invasive Surgery

wires for end effector control. The shaft can be rotated about its longitudinal axis. The end effector consists of an iron wrist and hand. The hand itself has two fingers that share one pivot axle. The axis of rotation of the wrist is arranged perpendicularly across from the finger axis. For our consideration we have chosen the following convention for axis assignment. The longitudinal axis of the shaft is conceived as z_M-axis. The rotational axis of the wrist constitutes the y_M-axis, while the hand rotation takes place about the x_M-axis. M indicates in this case, that these axes are parts of the mechanical system (see figure 3). Later we will introduce coordinate systems for the instruments that alleviate mathematical considerations. Note, that any correspondence of mechanical axes and this coordinate system only applies to the basic position of the instrument (i.e. all rotational angles are set to zero). Later we will explain how we exploit this initial congruence in order to reach the desired position of the end effector. The instruments are mounted on an aluminum adaptation that can be flanged to a Kuka robot. In addition, a second tube enwraping the shaft is fixed at this adaptation. It provides stability for the shaft and prevents the patient's body from contact with the rotating shaft. The robots bearing the instruments are standard industrial robots of the type Kuka KR6/2 (see figure 2). The last rotational axis of the Kuka robot and the rotational axis of the instrument are identical. Meaning that rotations about the longitudinal axis of the shaft can either be realized with the servos driving the instrument rotation or directly by the robot. We restrict our further considerations to one of the manipulators (instrument and robot). The kinematics of the other arm can easily be derived by symmetry.

1.2 Notation

Since we often deal with different coordinate systems and conversions between them, it is sensible to introduce some notation conventions. Within this text we will refer to coordinate systems with different capital letters. For example K denotes the coordinate frame of the Kuka robot. For transformations between

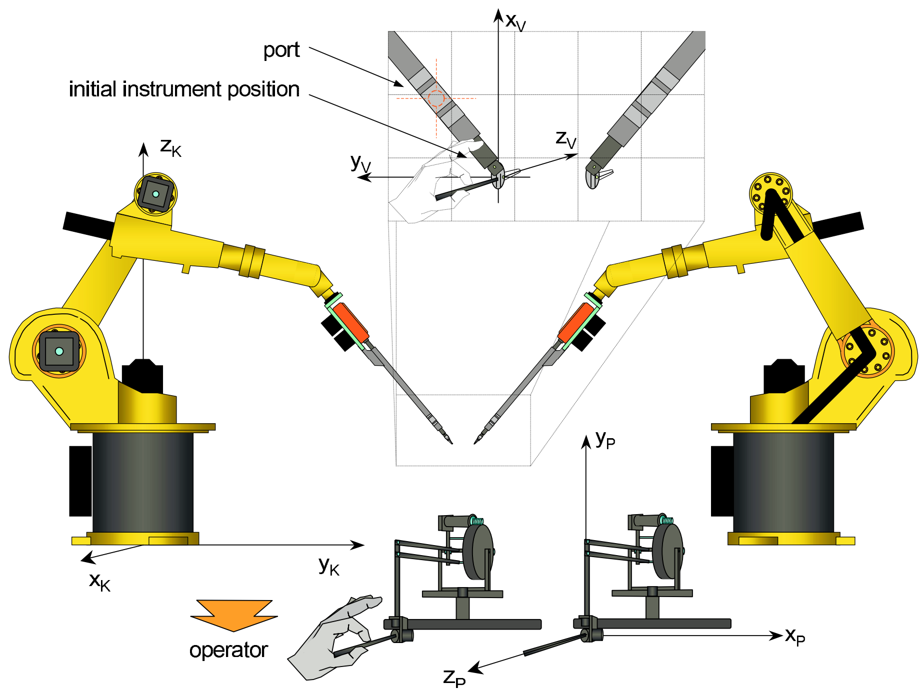

Fig. 2. System Overview

different system we use a capital T with the original system as superscript and the target system as subscript (both on the left side of the T). For example ${}^K_P T$ denotes the transformation of the Kuka system into the Phantom system. The rotational part of a transformation T_r is denoted as a 3×3-matrix, while translations T_t are indicated by subscripts with the corresponding axis. Example for a homogenous transformation matrix:

$$
{}^K_I T = \begin{pmatrix} {}^K_I T_{00} & {}^K_I T_{01} & {}^K_I T_{02} & {}^K_I T_x \\ {}^K_I T_{10} & {}^K_I T_{11} & {}^K_I T_{12} & {}^K_I T_y \\ {}^K_I T_{20} & {}^K_I T_{21} & {}^K_I T_{22} & {}^K_I T_z \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Vectors are represented by small letters, while points in 3D space are written down as capital letters. The corresponding coordinate system is indicated as a subscript. For example x_P means the x-axis of the Phantom system and O_K is the origin of the Kuka system. If the referred coordinate system is clear from the context, subscripts are omitted.

1.3 Coordinate Systems

In this section we want to describe some of the coordinate systems in detail which we will employ for our considerations. The most important issue in this

respect is the correlation between different systems. Because our working space is mostly specificated by the working space of the Kuka robots, we have chosen the Kuka system K as our base system. In our setup the user and the camera, respectively, is placed in front of the Kuka robots. Meaning the x_K-axis points in the direction of the user, while the z_K-axis points up to the ceiling (see figure 2). Since all systems have a right hand orientation, the y_K-axis points to the right hand side.
The Phantom is placed in front of the user and therefore its z_P-axis is collinear with the x_K-axis of the Kuka system. The y_P-axis points up to the ceiling, while the x_P-axis points to the right hand side (see figure 2).

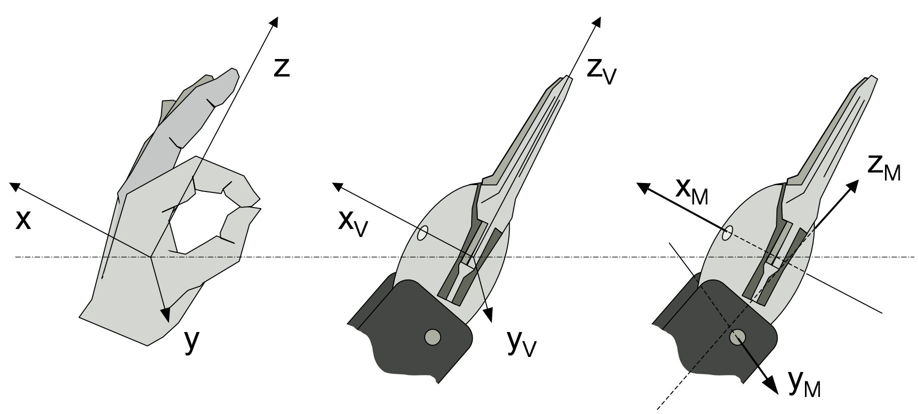

Fig. 3. Virtual Instrument Definition

A crucial issue is the determination of the coordinate system for the minimally invasive instrument. We want to design the control of the instrument as intuitive as possible. If we take the human hand as an archetype, we can make important observations. If we perform very precise manual tasks (e.g. a surgeon making a cut), we turn the hand about a rotation center that lies near the first link of the fingers (see figure 3 left side). On the one hand we want to include this behavior in our instrument control, on the other hand, we should preserve mechanical feasibility. A good compromise is shown in the middle of figure 3, where we have selected the mechanical rotation axis of the fingers to be the x_V-axis. Any rotation about this axis is called yaw. The z_V-axis (roll axis) points always in the direction of the closed end effector (e.g. gripper). Consequently, the y_V-axis (pitch axis) is aligned as shown in figure 3 in order to get a right hand system. Unfortunately, the mechanical axes do not intersect in the point we have chosen for the origin of the instrument system. Therefore we call the system established by the axes x_V, y_V and z_V virtual instrument system. Mechanical rotation axes of the real instrument are named x_M, y_M and z_M (see figure 3 right side). An important issue is the conversion of movements of the virtual instrument to real world axes rotations. We will refer to this later in detail.

1.4 Trocar Kinematics

The minimally invasive instruments are controlled by the operator via two Sensable Phantoms (6dof position control with 3dof force feedback). We want to impart the impression, that the user can control each instrument's motion as a direct mapping of the finger position (measured with the Phantom stylus; see figure 2). This behavior has already been implemented with commercially available systems like Intuitive's daVinci. This issue gets even more complicated if fixed ports for the instruments are demanded. A port is a small incision in the patient's body surface which render the surgeon possible to reach positions inside the body without making an invasive cut. In most of the cases three ports are needed (see figure 4). Two for the instruments and one for the endoscopic camera. It is clear that the possible movement of the instrument's shaft is restricted to insertion, retraction and rotation exclusively about the corresponding port. Every other motion (e.g. translation tangential to the body surface) would force the shaft to displace the port and harm the patient. This would be contradictory to the idea of minimally invasive surgery. Therefore the shaft axis of each instrument has always to be aligned with the corresponding port point.

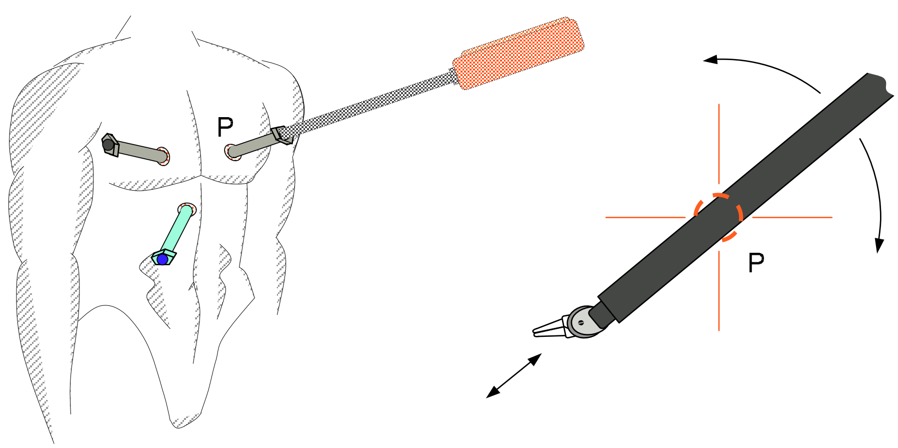

Fig. 4. Location of the Instrument and Camera Port

The most comfortable way of moving a surgical instrument inside the body would be cartesian control. Therefore we assume that the position of the instrument's end effector (e.g. gripper, scissors etc.) is given by a homogeneous transformation matrix. The position and rotation of the instrument and the position of the port has to be determined in our base system (system of the Kuka robot). That can easily be done for the port, but requires a coordinate transformation from the Phantom system towards the Kuka system ${}^{P}_{K}T$. As one can derive from the corresponding coordinate systems in figure 2, the z_P-axis of the phantom is collinear with the x_K-axis of the Kuka system. Correspondingly the x_P-axis of the phantom is collinear with the y_K-axis, while its y_P-axis points in the same direction like the z_K-axis of the base system. As we mentioned above, the z_V-axis of the instrument is always aligned with the end effector and points in its direction. On the other hand, the stylus of the Phantom points in the direction of the phantom's z_P-axis (see figure 2). That means, if we use the stylus as surrogate for a surgical instrument, its z_P-axis points in the reverse direction of the end effector. Therefore we additionally rotate the homogeneous transformation matrix of the stylus by 180 degrees. Meaning if the user holds the stylus in home position - i.e. the user's fingers point in direction of $-x_K$ - the fingers of the controlled instrument also point in direction of $-x_K$. This gives the user the impression of manually controlling the instrument's end effector (as depicted in figure 2). As a result, we get the transformation matrix ${}^{K}_{V}T$, for converting stylus movements (i.e. user input for the position of the virtual instrument system) to the Kuka system.

$$ {}^{K}_{V}T = {}^{K}_{P}T \cdot {}^{P}_{V'}T \cdot {}^{V'}_{V}T $$

$$ {}^{K}_{V}T = \begin{pmatrix} 0 & 0 & 1 & V_x^* \\ 1 & 0 & 0 & V_y^* \\ 0 & 1 & 0 & V_z^* \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{P}_{V'}T_{00} & {}^{P}_{V'}T_{01} & {}^{P}_{V'}T_{02} & {}^{P}_{V'}T_x \\ {}^{P}_{V'}T_{10} & {}^{P}_{V'}T_{11} & {}^{P}_{V'}T_{12} & {}^{P}_{V'}T_y \\ {}^{P}_{V'}T_{20} & {}^{P}_{V'}T_{21} & {}^{P}_{V'}T_{22} & {}^{P}_{V'}T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

${}^{K}_{P}T$ describes the renaming of the corresponding axes, where V_t^* is the displacement of the initial instrument system from the origin of the Kuka-system to the home position of the instrument. This home position is shown in figure 2. Transformation matrix ${}^{V'}_{V}T$ applies a rotation by 180 degrees about the y_P-axis of the phantom, while ${}^{P}_{V'}T$ gives us the measured user-applied position of the stylus (i.e. the virtual instrument). We now have both, the transformation of the instrument and the port, given in base (Kuka) coordinates.

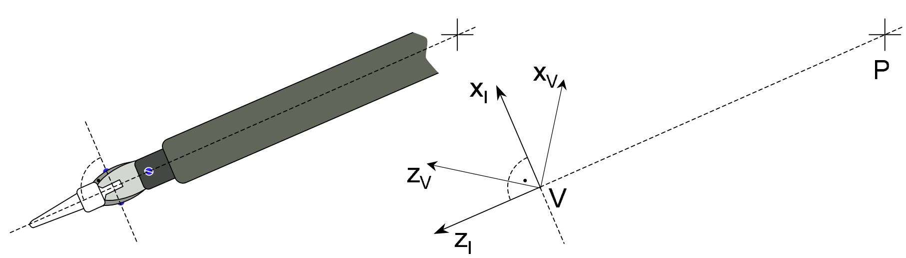

Fig. 5. Initial Position of the Virtual Instrument

As a next step, we determine the transformation matrix for a so-called initial instrument system I. This system is a hypothetical construct to facilitate the calculation of the backward kinematics of the instrument. The alignment of this initial position is determined by the positions of port and instrument (see figure 5). Point V marks the origin of the virtual instrument system, i.e. $V = {}^{K}_{V}T_t$. That means the z_I-axis of the virtual flange is identical with the vector $\overrightarrow{PV}$. We also demand that the y_I-axis of this system is parallel to the xy_K-plane of the Kuka system. The idea is, that we can easily calculate the homogenous transformation of this system in relation to the Kuka system. After that, the system I can be transferred to the virtual instrument system by subsequent rotations. But first of all, we describe the rotation of system I, relative to the base (Kuka) system, by Z-Y-X-Euler angles.

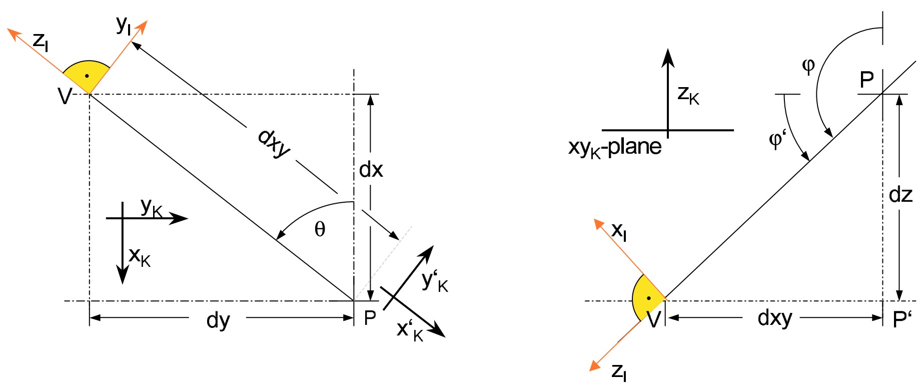

Fig. 6. Z-Y-(X)-Euler Angles of the Virtual Instrument System

We denote the angle of rotation about the initial z_K-axis of the Kuka system as θ. It can be determined with the $atan2$-function, which serves in robotics as an substitute for the well-known tan^{-1}-function. We measure the distance between the port and the instrument position by the difference vector $\vec{P} - \vec{V}$. The left side of figure 6 shows the first quadrant according to the sign of the differences. That means $atan2(dy, dx)$ will always calculate the angle θ, that results from a counterclockwise rotation of the negative x_K-axis until it points in direction of $\overrightarrow{PV}$. As a result we get a rotated intermediate system with axes x'_K,y'_K and $z'_K = z_K$. In order to get the desired initial position, we additionally have to rotate the intermediate system about the y'_K-axis. We calculate the length of dxy by Pythagoras' law: $dxy = \sqrt{dx^2 + dy^2}$. Therefore dxy will always become positive. The distance in z_K-direction is the z-component of the difference vector $\vec{P} - \vec{V}$. By employing the $atan2$-function, we get the angle φ'. Remember, since dxy is always positive, $atan2(dz, dxy)$ always returns the acute angle between $\overrightarrow{PF}$ and the xy_K-plane. We want to rotate z'_K, which is perpendicularly aligned to the xy_K-plane, until it is collinear with z_I. Therefore we have to add 90° to get the rotation angle φ. Because the y'_K-axis points into the drawing plane of figure 6 (right side), we have to apply $-\varphi$ for rotation (right hand rotation). As explained above, φ always lies in between 0° and 180°. This makes sense, because we do not want the instrument to get into a reverse constellation (reverse means x_V pointing downward). In order to keep y''_K $(= y'_K)$ parallel to the xy_K-plane, we set the rotation about the x''_K-axis to zero. Now we have reached the initial instrument position I: $x''_K = x_I; y''_K = y_I; z''_K = z_I$. Given these angles, we now can arrange the homogenous rotation matrix ${}^K_I T$ that transforms the Kuka-system to the initial position of the instrument.

$$
\begin{aligned}
{}^K_I T_{00} &= \cos(\theta) * \cos(\varphi) \\
{}^K_I T_{01} &= -\sin(\theta) \\
{}^K_I T_{02} &= \cos(\theta) * \sin(\varphi) \\
{}^K_I T_x &= V^*_x + V_x \\
{}^K_I T_{10} &= \sin(\theta) * \cos(\varphi) \\
{}^K_I T_{11} &= \cos(\theta) \\
{}^K_I T_{12} &= \sin(\theta) * \sin(\varphi) \\
{}^K_I T_y &= V^*_y + V_y \\
{}^K_I T_{20} &= -\sin(\varphi) \\
{}^K_I T_{21} &= 0 \\
{}^K_I T_{22} &= \cos(\varphi) \\
{}^K_I T_z &= V^*_z + V_z
\end{aligned}
$$

The derivation of this formula can be found in every book about robotics (e.g. [4] or [5]). Therefore we abstain from details here. Note that the position of the origin is the same for both, the virtual instrument system and the initial instrument system I (${}^K_I T_t = {}^K_V T_t$. This position is composed of an offset part ($V*_t$) (see above) and the user input for the instrument's position (V_t). On the other hand, the rotations of both systems are different. While the rotational part of ${}^K_I T$ gives the position of the instrument with no rotations applied to the instrument's axes (only to Kuka axes!), ${}^K_V T$ also determines the desired rotation of the instrument's axes. In order to apply this rotation mechanically, we need to know the transformation from ${}^K_I T$ (initial position) to ${}^K_V T$ (desired position). In other words we have to calculate T^I_V. This is best be done by taking the inverse matrix of the description of the initial position: ${}^K_I T^{-1} =^I_K T$. Now we get ${}^I_V T$ by a simple matrix multiplication: ${}^I_V T = {}^I_K T \cdot {}^K_V T$. With the help of this result, we can determine the X-Y-Z-Yaw-Pitch-Roll rotation angles for the instrument.

$$
\begin{aligned}
\alpha_V &= atan2({}^I_V T_{21}, {}^I_V T_{22}) \\
\beta_V &= atan2(-{}^I_V T_{20}, \sqrt{{}^I_V T_{00}^2 + {}^I_V T_{10}^2})
\end{aligned}
$$

$$\gamma_V = atan2({}^I_V T_{10}, {}^I_V T_{00})$$

We want to give a brief explanation wy we have chosen a X-Y-Z-Yaw-Pitch-Roll rotation. This order of angles can be directly realized with the mechanical structure of the instruments. The first rotation about the x_I-axis means that the instrument in its initial position makes a movement of the fingers. Note that the latter rotation of the instrument leaves the positions of all mechanical axes unchanged. This is very helpful, since Yaw-Pitch-Roll angles refer to a fixed system. As a next step we apply the rotation about the y_I-axis, by bending the wrist of the instrument. Now the rotation axis of the finger has been changed, but since we have already applied the corresponding rotation, it does not matter. The remaining rotation about the current z_I-axis is performed by turning the shaft. Every other order of angles will either have no mechanical correspondence (e.g. Z-Y-Z) or will lead to errors due to the fact that already executed rotations are changed afterwards due to mechanical dependencies. For example assume we had extracted X-Y-Z-Euler angles from the matrix. If we now perform the rotation about the x-axis, the y-axis changes its location to y' and the z-axis to z', respectively. But, e.g., the location of the mechanical y-axis stays the same. Therefore a subsequent rotation about the y'-axis cannot be performed mechanically.

In the last section we have determined the rotation angles of the virtual instrument from its initial position to the desired transformation. As mentioned above, we have assumed that all rotation axes of the virtual instrument intersect in one point. We now have to consider how to simulate this behavior with the real instrument and its mechanical axes. The mechanical x_M-axis and the virtual x_V-axis are identical as we have mentioned above. That means we can directly apply the x_V-rotation of the virtual instrument to its mechanical counterpart, the yaw of the fingers ($\alpha_M = \alpha_V$).
Because the x_M and y_M-axis of the instrument do not intersect, we cannot apply the y_V-rotation without modifications. We have to calculate it from known geometric features. We know the rotation angle about the y_V-axis of the virtual instrument (β_V). In addition we know, that the shaft of the instrument always has to pass the port P. Since virtual and mechanical y-axis are always parallel (but not necessarily identical), we can calculate the rotation about the mechanical y_M-axis (β_M). Note that the position of ${}^K_I T_t$ remains unchanged, but the shaft of the instrument is tilted about the port.

As one can derive from figure 7, we have given two sides of the triangle, where w is the length of the instrument's wrist and d is the distance between the port and the position of the virtual instrument system $V = {}^K_V T_t$. Additionally we have given the rotation angle $\beta_\triangle = \beta_V$ which is constituted by w and d. Therefore the following equations hold true:

$$\tan\left(\frac{\gamma_\triangle - \alpha_\triangle}{2}\right) = \frac{d-w}{d+w}\cot\left(\frac{\beta_\triangle}{2}\right), \quad \frac{\gamma_\triangle + \alpha_\triangle}{2} = 90^\circ - \frac{1}{2}\beta_\triangle;$$

By "atan"ing the first equation and adding the second one, we get:

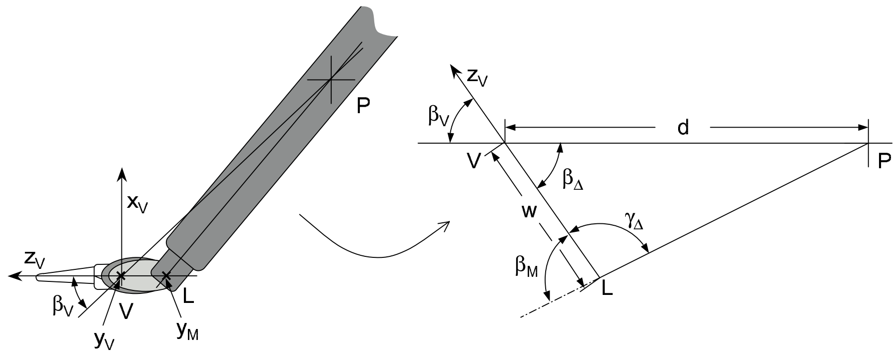

Fig. 7. Calculation of β_M

$$\gamma_\triangle = \tan^{-1}\left(\frac{d-w}{d+w}\cot\left(\frac{\beta_\triangle}{2}\right)\right) + 90^\circ - \frac{1}{2}\beta_\triangle;$$

We now get $\beta_M = 180^\circ - \gamma_\triangle$. Note that we will use this procedure regardless of the sign of β_V. We always use the absolute value of β_V within the formula explained above, but we remember the sign and apply it also to the resulting angle β_M.
The last angle to determine is the mechanical rotation of the instrument's shaft. The initial transformation of the shaft is set by the transformation of the robot flange F. In order to reach the desired position of the virtual instrument, we perform a rotation about the z_I-axis (see derivation of transform ${}^I_V T$). On the other hand, we only can apply rotations about the real z_M-axis, which is identical with the z_F-axis (see figure 8). It is clear from picture 7 that the longitudinal axis of the shaft z_M ($\overrightarrow{PL}$) is not collinear with the z_I-axis ($\overrightarrow{PV}$) as long as β_M is different from zero. In this case, we have to determine γ_M by an additional calculation. Therefore we need to find the positions of the mechanical instrument axes (esp. the shaft axis $z_M = z_F$) relative to the Kuka-system. Our chosen convention for the y_F-axis is, that it is initially parallel to the xy_K-plane. Note that every other convention would naturally change γ_M, but will not influence the final position of the instrument. In figure 8, the point V^+ is the position of the virtual instrument if γ_M were zero. Therefore we have to rotate the shaft until V^+ overlaps with the actual position of the virtual instrument $V = {}^K_V T_t$. In other words, we have to find a rotation that transforms x_F, which is parallel to a plane spanned by z_K and $\overrightarrow{PL}$, to x_I, which is parallel to a plane spanned by $\overrightarrow{PL}$ and $\overrightarrow{PV}$ (see figure 8). Therefore we first have to find the position of the inflexion point L (also see figure 7). We know the transformation of the virtual instrument ${}^K_V T$. Since the x_V- and the x_M-axis are identical, we can get the orientation of the wrist ${}^K_W T_r$, by back-rotating ${}^K_V T$ by $-\alpha_M$. Because the z-axis

of the resulting system points in the direction of the wrist, we now can step back along it for the known length of the wrist w. Thus altogether we get:

$$\vec{L} = {}^{K}_{V}T \cdot {}^{V}_{W}T \cdot \begin{pmatrix} 0 \\ 0 \\ -w \\ 1 \end{pmatrix}$$

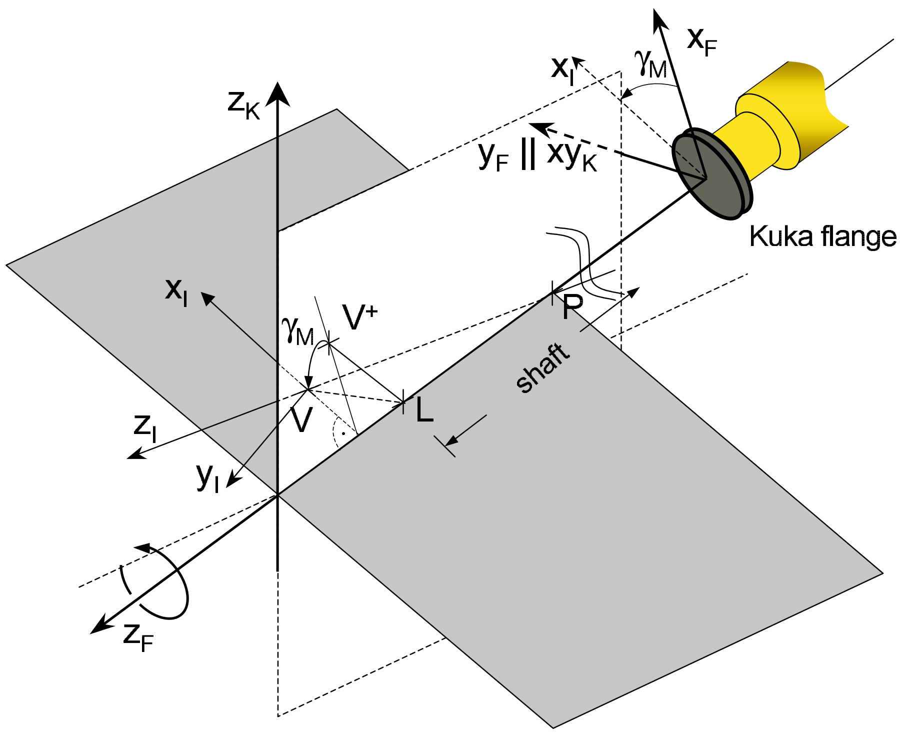

Fig. 8. Calculation of γ_M

Given all points, we now can assess γ_M. The best way to do this is calculating the intersection angle of the plane spanned by V^+, L and P, and the plane spanned by V, L and P. We get this angle by intersecting the normals of these planes. The plane spanned by V^+, L and P can be constituted by the cross product of the vector $\overrightarrow{PL}$, which is identical with the rotation axis of the instrument's shaft z_F, and the z_K-axis. The normal of the other plane spanned by the destination points of the instrument (V, L and P) is determined by the cross product of the vectors $\overrightarrow{PV}$ and $\overrightarrow{PL}$. Note that the direction and order of vectors for the cross product is crucial, in order to get the right direction of the normals. The angle between the normals can be calculated with the formula for the scalar product. Since the result will always lie in between 0° and 180°, we also have to regard the right quadrant, which can be derived from the knowledge of γ_V. Now we have all angles to control the instrument.

The next step is to determine the position and rotation of the flange of the Kuka robot (${}^K_F T$), where the instrument is fixed to. The orientation of the Kuka flange has to be the same like the initial orientation of the instrument's shaft. This is not identical with the initial rotation of the virtual instrument! The instrument should be positioned in an initial posture, i.e. with no shaft rotation. In this case, as mentioned above, the z_M-axis of the instrument is collinear with the vector $\overrightarrow{PL}$ while the y_M-axis is parallel to the xy_K-plane. Since the points P and L are given in coordinates of the Kuka system, we can directly derive the necessary rotation angles for the transformation ${}^K_F T$, which describes the position of the robot flange in the Kuka system. This is done in a similar fashion like for the initial position of the virtual instrument (see figure 4). In contrast to the arrangement of ${}^K_I T$, we do not use the formula for Z-Y-X-Euler angles here, but Z-Y-Z-Euler angles. Like the y_I-axis, the y_M-axis is initially parallel to the xy_K-plane. Therefore the last rotation is always zero, regardless if it were about the x''- or z''-axis. But taking the formula for Z-Y-Z-Euler angles has the advantage, that we can choose to perform a part (or even the complete) instrument rotation with the help of the last axis of the robot. In other words: $\gamma_M = \gamma'_M + \gamma_K$.
Once we have identified the flange rotation, we can simply find out the position of the flange point, by stepping back for the length l of the shaft in negative z_F-direction, starting at the inflexion point L. As we have explained above, L marks the link between wrist an shaft. If we take α_K for the z-rotation, β_K for the y'-rotation and γ_K for the z''-rotation, we get the following transform matrix ${}^K_F T$:

$$
\begin{aligned}
{}^K_F T_{00} &= \cos(\alpha_K) * \cos(\beta_K) * \cos(\gamma_K) - \sin(\alpha_K) * \sin(\gamma_K) \\
{}^K_F T_{01} &= -\cos(\alpha_K) * \cos(\beta_K) * \sin(\gamma_K) - \sin(\alpha_K) * \cos(\gamma_K) \\
{}^K_F T_{02} &= \cos(\alpha_K) * \sin(\beta_K) \\
{}^K_F T_x &= -{}^K_F T_{02} * l + L_x \\
{}^K_F T_{10} &= \sin(\alpha_K) * \cos(\beta_K) * \cos(\gamma_K) + \cos(\alpha_K) * \sin(\gamma_K) \\
{}^K_F T_{11} &= -\sin(\alpha_K) * \cos(\beta_K) * \sin(\gamma_K) + \cos(\alpha_K) * \cos(\gamma_K) \\
{}^K_F T_{12} &= \sin(\alpha_K) * \sin(\beta_K) \\
{}^K_F T_y &= -{}^K_F T_{12} * l + L_y \\
{}^K_F T_{20} &= -\sin(\beta_K) * \cos(\gamma_K) \\
{}^K_F T_{21} &= \sin(\beta_K) * \sin(\gamma_K) \\
{}^K_F T_{22} &= \cos(\beta_K) \\
{}^K_F T_z &= -{}^K_F T_{10} * l + L_z
\end{aligned}
\tag{1}
$$

We will use this matrix as an input for the inverse kinematics of the Kuka robot. This procedure will be described in the next section.

1.5 Kuka Inverse Kinematics

Since we have no possibility for a low-level cartesian control of the Kuka robots, but only control over the joint angles, we have to determine the inverse kinematics of the Kuka robots. That means, given the homogeneous transform matrix of the robot's flange, we have to find a mapping to extract the joint angles. This is best be done by addressing the first three degrees of freedom separately.

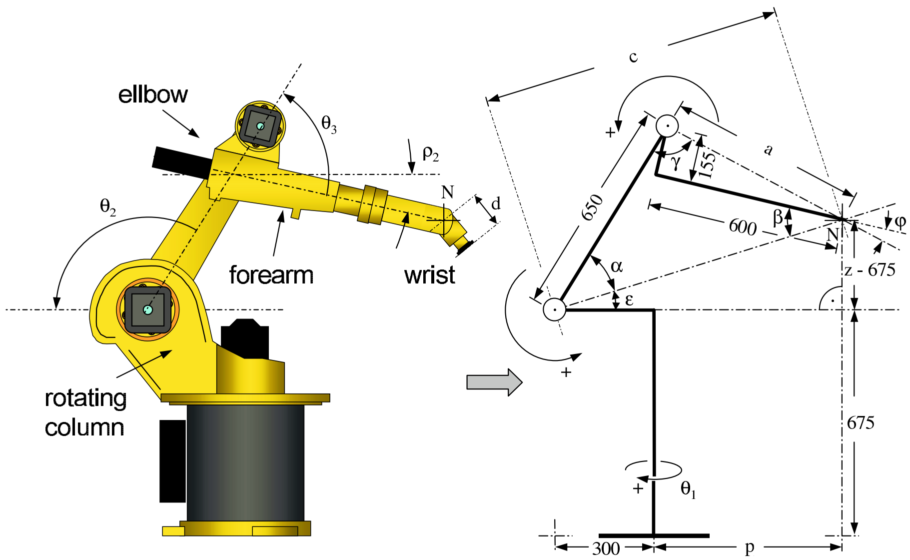

Fig. 9. Kuka Robot Geometry

The joint angles of the first three hinges uniquely determine the position of the point N. Note that there exists no uniquenes for the other direction, since the position of N can be reached by several joint angle configurations: We can choose to turn the rotating column by 180°. In addition, we can either put the arm in elbow up or elbow down posture. For our setup it was most convenient to turn the rotating column by 180° and move the elbow down. This can be seen in figure 9. It looks like the elbow were moved up, but that comes from the 180°-turn of the rotating column! We have given the transform of the robot's flange. In order to get the position of N we have to move back in z_F-direction for the length d of the last link (see figures 8 and 9). Consequently we get:

$$N = {}^{K}_{F}T \cdot \begin{pmatrix} 0 \\ 0 \\ -d \\ 1 \end{pmatrix}$$

$$\begin{aligned} N_x &= -d \cdot {}^{K}_{F}T_{02} + {}^{K}_{F}T_x \\ N_y &= -d \cdot {}^{K}_{F}T_{12} + {}^{K}_{F}T_y \\ N_z &= -d \cdot {}^{K}_{F}T_{22} + {}^{K}_{F}T_z \end{aligned} \tag{2}$$

Given the position of N, we now can derive the value of the first three joint angles $(\theta_1, \theta_2, \theta_3)$. The angle which is easiest to determine is θ_1. It can be calculated by:

$$\theta_1' = atan2(N_y, N_x)$$

As mentioned above, we additionally want to turn the rotating column by 180°. Therefore, in order to get θ_1 we add 180° if $\theta_1' < 0$, otherwise we subtract 180°. A dangerous situation will occur when the user wants to move the flange through the first or fourth quadrant of the xy_K-plane. Then the first joint, and with it the whole construction, will flip-over from $-180°$ to $180°$! We have to regard this issue when implementing the control software for the robot. Angles θ_2 and θ_3 are harder to figure out. We need a detailed plan of the robot's geometry (see figure 9 right side: all measurements are given in millimeters). First of all we identify the angle ϵ:

$$\epsilon = atan2(z - 675, p + 300); \;\; p = \sqrt{N_x^2 + N_y^2}$$

For calculating α and γ we will employ the three-side formula for triangles with oblique angles. Therefore we first need to know all three sides (a, b and c) of the triangle. One side is already predefined: $b = 650$mm. The others can be found as follows:

$$a = \sqrt{155^2 + 600^2}; \;\; c = \sqrt{(N_z - 675)^2 + (p + 300)^2};$$

Additionally we introduce s (the half of the triangles outline) and the radius of the inscribed circle r:

$$s = \frac{a + b + c}{2}; \;\; r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Now we can determine the interesting angles of this triangle:

$$\alpha = 2\tan^{-1}\left(\frac{r}{(s-a)}\right); \;\; \gamma = 2\tan^{-1}\left(\frac{r}{(s-c)}\right);$$

Because θ_2 is measured between upper arm and forearm, and not between upper arm and line a, we need the additional correction angle $\varphi = \tan^{-1}\left(\frac{155}{600}\right)$. Given all these angles we now can combine them to θ_2 and θ_3:

$$\theta_2 = \alpha + \epsilon - 180°; \;\; \theta_3 = \gamma + \varphi - 180°$$

Note the direction of rotation indicated by "+"-signs in figure 9. There is no danger of any flip-over for the second and the third joint, because θ_2 and θ_3 always lie in between 0° and −180°.
In order to determine the remaining angles, we need to know the transformation matrix of the forearm in point N. We get this homogeneous transformation matrix ${}^K_A T$ by rotating the base system about its z_K-axis by $\rho_1 = -\theta_1$. For orientation of the base system see 2. This gives use the transform of the base system after application of θ_1 (Note that the negative sign is due to the left-hand rotation of the first joint). After that we turn the z'_K-axis, which still points to the ceiling, into the direction of the forearm. As one can derive from figure 9, this can be done by rotating $\rho_2 = \theta_2 + \theta_3 + 90°$ about the y'_K-axis. Given ρ_1 and ρ_2, we can determine ${}^K_A T$ in a similar fashion like we did for ${}^K_I T$ and ${}^K_F T$. We use again the formula for arranging a homogenous transform matrix from Z-Y-X-Euler angles. In our case, the rotation about the x''_K-axis is zero.

$$
\begin{aligned}
{}^K_A T_{00} &= \cos(\rho_1) * \cos(\rho_2);\\
{}^K_A T_{01} &= -\sin(\rho_1);\\
{}^K_A T_{02} &= \cos(\rho_1) * \sin(\rho_2);\\
{}^K_A T_{x} &= N_x;\\
{}^K_A T_{10} &= \sin(\rho_1) * \cos(\rho_2);\\
{}^K_A T_{11} &= \cos(\rho_1);\\
{}^K_A T_{12} &= \sin(\rho_1) * \sin(\rho_2);\\
{}^K_A T_{y} &= N_y;\\
{}^K_A T_{20} &= -\sin(\rho_2);\\
{}^K_A T_{21} &= 0;\\
{}^K_A T_{22} &= \cos(\rho_2);\\
{}^K_A T_{z} &= N_z;
\end{aligned}
$$

For further steps we need to determine the orientation of the flange in relation to the orientation of the forearm. In other words we are looking for ${}^A_F T$. The easiest way to do this, is multiplying the inverse of the forearm transformation with the flange transformation: ${}^K_A T^{-1} \cdot {}^K_F T = {}^A_K T \cdot {}^K_F T = {}^A_F T$.

Note that after applying ${}^K_A T$, the z_A-axis of the forearm system points in the distal direction of the forearm. The rotation axes of the last three joints intersect in one point (see figure 10). In their initial orientation, the rotation axis of the fourth and sixth joint are identical to z_A, while the fifth one is identical to

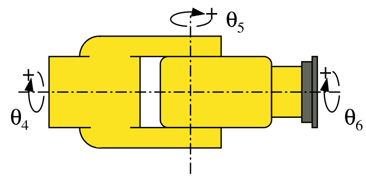

Fig. 10. Top View of the Wrist

y_A. Therefore we can extract the corresponding angles from the transformation matrix with the rules for Z-Y-Z-Euler angles:

$$\theta_4 = -atan2({}^A_F T_{12}, {}^A_F T_{02});$$
$$\theta_5 = atan2(\sqrt{{}^A_F T_{20}^2 + {}^A_F T_{21}^2}, {}^A_F T_{22});$$
$$\theta_6 = -atan2({}^A_F T_{21}, -{}^A_F T_{20});$$

The negative signs for θ_4 and θ_6 come from the left-handed rotation of the corresponding mechanical axes! Due to the fact that we operate the robot in a "headlong" position, we have chosen a configuration with a reverted wrist in order to usually get smaller angles. Therefore we additionally turn the fourth and sixth joint by 180° and drive θ_5 in the opposite direction. Now we have the complete information to control our manipulator system. We have got three rotation angles for the instrument and six joint angles for the Kuka robot.

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